TMUA

Test of Mathematics for University Admission

Paper 1: Applications of Mathematical Knowledge

Mock paper-01

Time allowed:	75 minutes
Questions:	20 multiple-choice
Calculator:	not permitted
Formula booklet:	not provided
Generated:	2026-05-03

Instructions

- Each question has between 4 and 8 lettered options. Exactly one option is correct.
- Mark your chosen letter for each question. There is no penalty for incorrect answers.
- All quantities are exact; no decimal approximations are required.
- The answer key and worked solutions appear at the end of this paper.

Questions

1. Given that $x + \frac{1}{x} = 3$, the value of $x^2 + \frac{1}{x^2}$ is

A 3

B 6

C 7

D 9

E 11

2. What is the value of

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + 19^2 - 20^2\,?$$

A -230

B -210

C -10

D 10

E 210

F 230

3. The point $(3, k)$ lies on the circle with centre $(1, 2)$ and radius $\sqrt{20}$. The sum of the possible values of k is

A -12

B -4

C 4

D 6

E 8

F 12

4. Given that $\sin\theta - \cos\theta = \frac{1}{2}$, the value of $\sin\theta\cos\theta$ is

A $-\frac{3}{8}$

B $-\frac{1}{8}$

C $\frac{1}{4}$

D $\frac{3}{8}$

E $\frac{1}{2}$

F $\frac{3}{4}$

5. The functions f and g are defined for all real x by $f(x) = x + 3$ and $g(x) = x^2$. Find all real values of x for which $f(g(x)) = g(f(x))$.

A $x = -3$

B $x = -2$

C $x = -1$

D $x = 1$

E no real solutions

6. Given that $\log_a b = 2$ and $\log_b c = 3$, the value of $\log_c(ab^2)$ is

A $\frac{1}{6}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{5}{6}$

E 5

F 12

7. The 2nd, 4th and 8th terms of an arithmetic progression form a non-constant geometric progression. What is the common ratio of this geometric progression?

A $\frac{1}{2}$

B $\frac{4}{3}$

C 2

D 3

E 4

8. The curve $y = x^3 + ax^2 + bx$ has a stationary point at $x = 1$, and the tangent to the curve at $x = 0$ has gradient -9. What is the y-coordinate of the other stationary point of the curve?

A -5

B 11

C 27

D 54

9. The curve $y = f(x)$ has a single vertical asymptote at $x = 2$ and a single horizontal asymptote at $y = -1$. The curve $y = g(x)$ is obtained by the following sequence of transformations applied to $y = f(x)$: a translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, then a reflection in the x-axis, then a stretch parallel to the y-axis with scale factor 2. What are the asymptotes of $y = g(x)$?

A $x = -1$ and $y = 2$

B $x = -1$ and $y = -2$

C $x = -1$ and $y = 1$

D $x = 5$ and $y = 2$

E $x = 5$ and $y = -2$

F $x = 1$ and $y = 2$

G $x = 1$ and $y = -2$

10. The polynomial $p(x) = x^3 + ax^2 + bx + 12$ has $(x - 2)$ as a factor, and leaves a remainder of -18 when divided by $(x + 1)$. What is the value of $a + b$?

A -29

B -13

C 3

D 16

E 29

11. The total area of the finite region(s) enclosed between the curve $y = x^3 - 4x$ and the x-axis is

A 0

B 4

C 8

D 16

E 24

12. The function $f(\theta) = 4\sin\theta + 3\cos\theta$ attains its maximum value at $\theta = \alpha$, where $0 < \alpha < \frac{\pi}{2}$. The value of $\tan(2\alpha)$ is

A $-\frac{24}{7}$

B $-\frac{7}{24}$

C $\frac{7}{24}$

D $\frac{8}{3}$

E $\frac{24}{25}$

F $\frac{24}{7}$

13. Let a and b be positive real numbers. Consider the following three statements.

I. If $a + b \geq 2\sqrt{ab}$, then $a = b$.

II. $a^2 + b^2 \geq 2ab$.

III. If $\frac{1}{a} + \frac{1}{b} \leq \frac{4}{a+b}$, then $a = b$.

Which of the statements are true for all positive real a and b?

A None of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

14. In the expansion of $(1 + ax)^n$, where a is a non-zero real constant and n is a positive integer, the coefficient of x^2 is 60 and the coefficient of x^3 is 160. What is the value of $a + n$?

A 4

B 6

C 8

D 12

15. The function f is defined for all real x by

$$f(x) = |x^2 - 6x + 5|.$$

The equation $f(x) = k$ has exactly four distinct real solutions. Find the complete set of values of k.

A $0 < k < 4$

B $0 \leq k \leq 4$

C $0 < k \leq 4$

D $1 < k < 5$

E $0 < k < 5$

F $-4 < k < 4$

G $0 < k < 6$

H $k > 0$

16. The line ℓ is tangent to the curve $y = x^4 - 8x^2 + 5x$ at two distinct points. The y-intercept of ℓ is

A -32

B -16

C -8

D -5

E -4

F 4

G 5

H 16

17. The equation

$$\log_2(x^2 + a) \;=\; 1 + \log_2(x + 1)$$

in the unknown x has exactly one real solution. Find the complete set of values of the real constant a for which this holds.

A $a = 3$

B $a \leq -1$

C $a < -1$ or $a = 3$

D $a \leq -1$ or $a = 3$

E $-1 \leq a \leq 3$

F $a \leq -1$ or $a \geq 3$

G $a < -1$ or $a > 3$

H $a = -1$ or $a = 3$

18. Let $f(x) = x^2 - x - 1$. How many distinct real values of x satisfy $f(f(x)) = x$?

A 0

B 1

C 2

D 3

E 4

F infinitely many

19. Evaluate

$$\int_0^2 \frac{x^3}{x^3 + (2 - x)^3}\, dx.$$

A 0

B $\frac{1}{2}$

C $\frac{2}{3}$

D 1

E $\frac{4}{3}$

F 2

20. Find the number of ordered pairs of integers (a, b) with $2 \leq a < b \leq 100$ for which $\log_a b$ is rational.

A 16

B 21

C 23

D 24

E 25

F 30

G 50

Answer key

Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	C	6	D	11	C	16	B
2	B	7	C	12	A	17	D
3	C	8	C	13	G	18	E
4	D	9	A	14	C	19	D
5	C	10	C	15	A	20	E

Worked solutions

Question 1

Square both sides: $\left(x+\frac{1}{x}\right)^2 = x^2 + 2\cdot x\cdot\frac{1}{x} + \frac{1}{x^2} = x^2 + \frac{1}{x^2} + 2$. So $x^2 + \frac{1}{x^2} = 3^2 - 2 = 7$.

Distractor analysis:

A Quotes the original value $x + 1/x = 3$ without performing any manipulation.

B Doubles instead of squaring, computing $2(x + 1/x) = 6$.

D Squares correctly but drops the $+2$ cross-term, getting $(x + 1/x)^2 = 9 = x^2 + 1/x^2$.

E Squares but adds the cross-term instead of subtracting it: $9 + 2 = 11$.

Question 2

Pair consecutive terms and use the difference of two squares: $(2k-1)^2-(2k)^2 = -(4k-1)$. Summing over $k = 1, 2, \ldots, 10$ gives $-\sum_{k=1}^{10}(4k-1) = -(4\cdot 55 - 10) = -210$.

Distractor analysis:

A Mis-applies the difference of two squares as $-(4k + 1)$, giving -230.

C Treats each pair as -1 (linear thinking, ignoring the squares).

D Same linear slip with sign reversed: $+1$ per pair, $+10$ total.

E Pairs the wrong way: $(2k)^2 - (2k-1)^2 = +(4k-1)$, summing to $+210$.

F Combines wrong-way pairing with the constant sign slip: $+230$.

Question 3

Substituting gives $(3-1)^2 + (k-2)^2 = 20$, so $(k-2)^2 = 16$ and $k = 2 \pm 4$, i.e. $k = 6$ or $k = -2$. Their sum is 4. (Equivalently, by symmetry in $y = 2$, the two values sum to $2\cdot 2 = 4$.)

Distractor analysis:

A Computes the product $6\cdot(-2) = -12$ instead of the sum.

B Sign slip when expanding, solving $(k + 2)^2 = 16$.

D Gives only the positive root $k = 6$.

E Adds the absolute values $|6| + |-2| = 8$.

F Computes the product magnitude $|6\cdot(-2)| = 12$.

Question 4

Square both sides: $(\sin\theta - \cos\theta)^2 = 1 - 2\sin\theta\cos\theta$. So $1 - 2\sin\theta\cos\theta = \frac{1}{4}$, giving $\sin\theta\cos\theta = \frac{3}{8}$.

Distractor analysis:

A Sign slip on the cross-term: writes $(\sin\theta - \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$.

B Forgets the $\sin^2\theta + \cos^2\theta = 1$ contribution, gets $-2\sin\theta\cos\theta = \frac{1}{4}$.

C Forgets to square the right-hand side: $1 - 2\sin\theta\cos\theta = \frac{1}{2}$.

E Misreads the LHS as $\sin\theta \cdot \cos\theta = \frac{1}{2}$ (minus sign as multiplication slip), copies the RHS.

F Drops the factor of 2 on the cross-term: $1 - \sin\theta\cos\theta = \frac{1}{4}$.

Question 5

$f(g(x)) = f(x^2) = x^2 + 3$ and $g(f(x)) = g(x+3) = (x+3)^2 = x^2 + 6x + 9$. Equating gives $x^2 + 3 = x^2 + 6x + 9$, so $6x = -6$ and hence $x = -1$.

Distractor analysis:

A Reads $f(g(x)) = g(f(x))$ as $f(x) = 0$, solving $x + 3 = 0$.

B Drops the factor of 2 in the cross-term of $(x+3)^2$, expanding it as $x^2 + 3x + 9$, then solves $3 = 3x + 9$.

D Correct algebra up to $6x = -6$, then a sign slip gives $x = 1$.

E Forgets the cross-term entirely, expanding $(x+3)^2$ as $x^2 + 9$, which gives $3 = 9$ and no solution.

Question 6

From $\log_a b = 2$, $b = a^2$, so $a = b^{1/2}$. From $\log_b c = 3$, $c = b^3$. Hence $ab^2 = b^{1/2} \cdot b^2 = b^{5/2}$, and since $\log_c b = \frac{1}{3}$, $\log_c(ab^2) = \frac{5}{2} \cdot \frac{1}{3} = \frac{5}{6}$.

Distractor analysis:

A Computes $\log_c a = \dfrac{1}{\log_a b \cdot \log_b c} = \dfrac{1}{6}$ but drops the $\log_c b^2$ term.

B Forgets the square on b, computing $\log_c(ab) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.

C Computes only $\log_c b^2 = 2\log_c b = \frac{2}{3}$, dropping the $\log_c a$ contribution.

E Adds the two given log values: $2 + 3 = 5$.

F Inverts both relations, reading $a = b^2$ and $b = c^3$, so $a = c^6$ and $ab^2 = c^{12}$.

Question 7

Let the AP have first term a and common difference d. The three terms are $a+d$, $a+3d$ and $a+7d$. For a GP, $(a+3d)^2 = (a+d)(a+7d)$. Expanding: $a^2 + 6ad + 9d^2 = a^2 + 8ad + 7d^2$, so $2d^2 = 2ad$, i.e. $d(d-a) = 0$. The GP is non-constant, so $d \neq 0$, giving $d = a$. The three terms become $2a$, $4a$, $8a$, so the common ratio is 2.

Distractor analysis:

A Computes the reciprocal u_2/u_4 instead of u_4/u_2.

B Misreads the positions as the 1st, 4th and 8th terms; solving $(a+3d)^2 = a(a+7d)$ gives $a = 9d$, yielding ratio $\frac{4}{3}$.

D Implicitly assumes $a = 0$ so the three terms are $d, 3d, 7d$, then reads the ratio $3d/d = 3$ without checking that the sequence is geometric.

E Takes the ratio $u_8/u_2 = 8a/2a$ but reports this single jump as the common ratio, forgetting it spans two steps.

Question 8

$y' = 3x^2 + 2ax + b$. The gradient at $x = 0$ equals b, so $b = -9$. The stationary point at $x = 1$ gives $3 + 2a + b = 0$, hence $a = 3$. Then $y' = 3x^2 + 6x - 9 = 3(x + 3)(x - 1)$, so the other stationary point is at $x = -3$. Substituting: $y(-3) = -27 + 27 + 27 = 27$.

Distractor analysis:

A Evaluates $y(1) = 1 + 3 - 9 = -5$ — the original stationary point, not the 'other' one.

B Factoring slip: writes $y' = 3(x + 1)(x - 3)$ instead of $3(x + 3)(x - 1)$, so the 'other' stationary point is at $x = -1$ with $y(-1) = -1 + 3 + 9 = 11$.

D Forgets the x^3 term when evaluating $y(-3)$, computing $3(-3)^2 - 9(-3) = 27 + 27 = 54$.

Question 9

The translation by $(-3, 0)$ replaces x by $x+3$, shifting the vertical asymptote from $x = 2$ to $x = -1$; the horizontal asymptote stays at $y = -1$. Reflection in the x-axis negates y, so $y = -1$ becomes $y = 1$ (vertical asymptote unchanged). The vertical stretch by factor 2 scales y, sending $y = 1$ to $y = 2$. Final asymptotes: $x = -1$ and $y = 2$.

Distractor analysis:

B Forgets the sign change from the reflection in the x-axis (treats the horizontal asymptote as multiplied only by the stretch factor).

C Applies the reflection but forgets to apply the vertical stretch to the horizontal asymptote.

D Translates the vertical asymptote in the wrong direction (adds 3 instead of subtracting), but handles the vertical chain correctly.

E Wrong-direction horizontal translation AND missed sign flip from the reflection.

F Computes $|2 - 3| = 1$ for the new vertical asymptote (absolute-value slip), with vertical chain correct.

G Same translation-magnitude slip as F, plus missed reflection sign flip.

Question 10

By the factor theorem, $p(2) = 0$: $8 + 4a + 2b + 12 = 0$, so $2a + b = -10$. By the remainder theorem, $p(-1) = -18$: $-1 + a - b + 12 = -18$, so $a - b = -29$. Adding: $3a = -39$, so $a = -13$ and $b = -10 - 2(-13) = 16$. Therefore $a + b = 3$.

Distractor analysis:

A Reports $a - b = -29$ instead of $a + b$.

B Stops after computing $a = -13$ and reports a alone.

D Reports $b = 16$ alone.

E Computes $b - a = 16 - (-13) = 29$ instead of $a + b$.

Question 11

Factorise: $y = x(x-2)(x+2)$, so the curve meets the x-axis at $x = -2, 0, 2$. It is positive on $(-2, 0)$ and negative on $(0, 2)$, so the enclosed region splits into two lobes whose areas must be summed in absolute value. With $F(x) = \frac{x^4}{4} - 2x^2$: $\int_{-2}^{0}(x^3 - 4x)\,dx = F(0) - F(-2) = 0 - (4 - 8) = 4$ and $\int_0^2(x^3 - 4x)\,dx = F(2) - F(0) = -4$. The integrand is odd, so the two lobes have equal area. Total area $= 4 + |-4| = 8$.

Distractor analysis:

A Integrates $\int_{-2}^{2}(x^3 - 4x)\,dx$ in one go without splitting at the roots; the odd integrand gives 0 and the student forgets that signed area cancels.

B Computes only one of the two symmetric lobes and forgets to add the other.

D Computes one lobe's area as 4, doubles to 8 (correctly) for symmetry, then doubles again under the impression that each lobe contributes 8.

E Uses the incorrect antiderivative $\frac{x^4}{4} - 4x^2$ (integrating $-4x$ to $-4x^2$ instead of $-2x^2$); each lobe then evaluates to 12, giving 24.

Question 12

Writing $f(\theta) = R\sin(\theta + \phi)$ with $R = 5$ and $\tan\phi = \frac{3}{4}$, the maximum occurs when $\theta + \phi = \frac{\pi}{2}$, i.e. $\tan\alpha = \cot\phi = \frac{4}{3}$. Then $\tan(2\alpha) = \dfrac{2\tan\alpha}{1 - \tan^2\alpha} = \dfrac{8/3}{1 - 16/9} = \dfrac{8/3}{-7/9} = -\dfrac{24}{7}$.

Distractor analysis:

B Computes $\sin(2\alpha) = \frac{24}{25}$ and $\cos(2\alpha) = -\frac{7}{25}$, then takes the ratio upside-down ($\cos/\sin$) to get $-\frac{7}{24}$.

C Same upside-down ratio as B with the sign of $\cos(2\alpha)$ also slipped.

D Forgets the denominator $1 - \tan^2\alpha$ in the double-angle formula and reports $2\tan\alpha = \frac{8}{3}$.

E Confuses $\tan(2\alpha)$ with $\sin(2\alpha) = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}$.

F Reads $\tan\alpha = \frac{3}{4}$ off the R-formula auxiliary angle ($\tan\phi$) instead of $\cot\phi = \frac{4}{3}$, giving $\frac{24}{7}$.

Question 13

Statement II rearranges to $(a - b)^2 \geq 0$, true for all real a, b. Statement III: $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, so the hypothesis becomes $(a + b)^2 \leq 4ab$, i.e. $(a - b)^2 \leq 0$, forcing $a = b$. Statement I is the trap: $a + b \geq 2\sqrt{ab}$ is the AM-GM inequality, true for ALL positive a, b — not only when $a = b$. So the hypothesis always holds but the conclusion need not (e.g. $a = 1, b = 4$ gives $5 \geq 4$ with $a \neq b$). Hence II and III only.

Distractor analysis:

A Doubts II by treating the strict-inequality case $a \neq b$ as a counter-example to a non-strict claim, and gets the algebra in III backwards.

B Confuses the AM-GM equality condition with the inequality itself: knows equality holds iff $a = b$ and wrongly concludes the $\geq$ version forces $a = b$ too.

C Spots II immediately but doesn't push through the algebra of III.

D Does the III manipulation correctly but doubts II for some sign reason and rules it out unnecessarily.

E Falls for the AM-GM equality trap in I, correctly accepts II, gives up on III.

F Falls for the AM-GM equality trap in I and correctly handles III, but mistakenly thinks II requires strict inequality to be 'meaningful'.

H Correctly handles II and III but also falls for the AM-GM equality trap in I.

Question 14

The coefficient of x^r in $(1+ax)^n$ is $\binom{n}{r}a^r$, so $\binom{n}{2}a^2 = 60$ and $\binom{n}{3}a^3 = 160$. Dividing the second by the first eliminates the awkward power of a: $\frac{n-2}{3}\,a = \frac{8}{3}$, hence $(n-2)a = 8$. Substituting $a = \frac{8}{n-2}$ into $\binom{n}{2}a^2 = 60$ gives $\frac{n(n-1)}{2} \cdot \frac{64}{(n-2)^2} = 60$, which simplifies to $7n^2 - 52n + 60 = 0$, i.e. $(7n-10)(n-6) = 0$. Since n is a positive integer, $n = 6$ and $a = 2$. (Check: $\binom{6}{2} \cdot 4 = 60$ and $\binom{6}{3} \cdot 8 = 160$.) So $a + n = 8$.

Distractor analysis:

A Solves $\binom{n}{2}a^2 = 60$ with $n = 6$ to get $a^2 = 4$ and takes $a = -2$ without checking the x^3 sign; then $a + n = -2 + 6 = 4$.

B Correctly extracts $n = 6$ but reports n alone, forgetting to add a.

D Solves correctly to $n = 6$, $a = 2$ but computes the product $an = 12$ instead of the sum.

Question 15

Let $g(x) = x^2 - 6x + 5 = (x-1)(x-5)$. The parabola has roots at $x = 1, 5$ and a minimum at $x = 3$, where $g(3) = -4$. Reflecting the portion below the x-axis gives f: it touches zero at $x = 1, 5$, rises to a local maximum of 4 at $x = 3$, and increases to $+\infty$ as $|x| \to \infty$. The horizontal line $y = k$ meets this graph in: 0 points if $k < 0$; 2 points if $k = 0$; 4 points if $0 < k < 4$; 3 points if $k = 4$ (the bump's peak counts once); 2 points if $k > 4$. So $f(x) = k$ has exactly four solutions iff $0 < k < 4$.

Distractor analysis:

B Includes both endpoints, missing that $k = 0$ gives only 2 solutions and $k = 4$ gives only 3.

C Includes $k = 4$, forgetting that at the local maximum the line is tangent to the bump (one solution, not two).

D Uses the roots of g as the bounds, confusing the x-values where $f = 0$ with the y-values that bound the four-solution region.

E Uses $f(0) = 5$ as the upper bound instead of the local maximum value $f(3) = 4$.

F Forgets the modulus and reads off the vertex value -4 as the lower bound, treating $g(x) = k$ rather than $|g(x)| = k$.

G Misreads the coefficient -6 as the height of the local maximum after reflection.

H Recognises that k must be positive but fails to find the upper bound.

Question 16

If $\ell : y = mx + c$ is tangent to $y = f(x)$ at $x = a$ and $x = b$, then $f(x) - (mx + c)$ has double roots at both points, so

$$x^4 - 8x^2 + 5x - (mx + c) = (x-a)^2(x-b)^2.$$

Expanding the RHS as $x^4 - 2(a+b)x^3 + [(a+b)^2 + 2ab]x^2 - 2ab(a+b)x + a^2b^2$ and matching coefficients: the x^3 term forces $a + b = 0$, so $b = -a$; the x^2 term gives $-2a^2 = -8$, so $a^2 = 4$; the x^1 term gives $5 - m = 0$, so $m = 5$; the constant term gives $-c = a^2b^2 = 16$, hence $c = -16$. (Check: tangent points $(2, -6)$ and $(-2, -26)$ both have $y' = 5$.)

Distractor analysis:

A Computes $f(2) + f(-2) = -6 + (-26) = -32$ (sum of y-coordinates of tangent points) and reports that as the intercept.

C Reads off the coefficient of x^2 in the original equation, mistaking it for the intercept.

D Finds the gradient $m = 5$ correctly but quotes $-m$ as the intercept.

E Solves $-2a^2 = -8$ to get $a^2 = 4$, then carries a stray sign through to call the intercept $-a^2 = -4$.

F Solves $a^2 = 4$ for the tangent abscissae and reports a^2 itself as the intercept.

G Confuses the gradient $m = 5$ with the intercept c.

H Sign error: writes $a^2b^2 = c$ instead of $-c$, giving $c = 16$.

Question 17

Rewrite RHS as $\log_2(2(x+1))$, so the equation becomes $x^2 + a = 2x + 2$, i.e. $x^2 - 2x + (a-2) = 0$, with domain $x > -1$. (The $x^2 + a > 0$ condition is automatic on solutions, since there $x^2 + a = 2(x+1) > 0$.) Roots: $x = 1 \pm \sqrt{3-a}$, real iff $a \leq 3$. Case 1: double root $a = 3$ gives $x = 1$, valid. Case 2: two distinct roots, exactly one in domain — the larger root $1 + \sqrt{3-a} > -1$ always; we need the smaller $1 - \sqrt{3-a} \leq -1$, i.e. $\sqrt{3-a} \geq 2$, i.e. $a \leq -1$. (At $a = -1$, the smaller root is -1, excluded by the strict $x > -1$.) So exactly one solution iff $a \leq -1$ or $a = 3$.

Distractor analysis:

A Spots the double root at $a = 3$ but overlooks that the smaller root is killed by the domain restriction for $a \leq -1$.

B Handles the domain-cull case but forgets the double-root case at $a = 3$.

C Uses strict inequality at -1, mishandling the boundary $a = -1$ where the spurious root is exactly -1.

E Inverts the logic: identifies the range where both roots lie in the domain and reports it.

F Treats $a \geq 3$ as giving one solution by confusing 'no real roots' for $a > 3$ with 'one solution'.

G Same error as F (treats $a > 3$ as one solution) and additionally uses strict $<$ at -1.

H Only checks the two boundary values where 'something special' happens, missing the entire ray $a < -1$.

Question 18

Any x with $f(x) = x$ also satisfies $f(f(x)) = x$, so $f(x) - x$ divides $f(f(x)) - x$. Now $f(x) = x$ gives $x^2 - 2x - 1 = 0$, with roots $x = 1 \pm \sqrt{2}$. Expanding, $f(f(x)) = x^4 - 2x^3 - 2x^2 + 3x + 1$, so $f(f(x)) - x = x^4 - 2x^3 - 2x^2 + 2x + 1$. Polynomial division by $x^2 - 2x - 1$ gives quotient $x^2 - 1$, hence

$$f(f(x)) - x = (x^2 - 2x - 1)(x^2 - 1).$$

The second factor contributes $x = \pm 1$, which form a 2-cycle of f (since $f(1) = -1$ and $f(-1) = 1$). All four roots are real and distinct.

Distractor analysis:

A Brute-forces the quartic, finds no rational roots immediately, and concludes there are no real solutions.

B Spots $x = 1$ as a root by inspection but doesn't pursue further factorisation.

C Solves only $f(x) = x$, getting $x = 1 \pm \sqrt{2}$, and forgets that $f(f(x)) = x$ also captures 2-cycles.

D Finds the two fixed points and one of ± 1, but slips when dividing the quartic and overlooks the other.

F Misreads $f(f(x)) = x$ as the statement that $f \circ f$ is the identity map, concluding every x is a solution.

Question 19

Let $I = \int_0^2 \frac{x^3}{x^3+(2-x)^3}\,dx$. Substituting $u = 2 - x$ (so $du = -dx$ and the limits swap) gives

$$I = \int_0^2 \frac{(2-u)^3}{(2-u)^3 + u^3}\,du.$$

Renaming u to x and adding to the original,

$$2I = \int_0^2 \frac{x^3 + (2-x)^3}{x^3 + (2-x)^3}\,dx = \int_0^2 1\,dx = 2,$$

so $I = 1$. (A direct attack expanding $(2-x)^3$ reduces the integrand to $\frac{x^3}{2(3x^2-6x+4)}$, requiring polynomial division and a logarithmic term — the symmetry pairing collapses it instantly.)

Distractor analysis:

A Notes that $f(x) - \frac{1}{2}$ is antisymmetric about $x = 1$ and concludes the integral is zero, forgetting to add back the $\frac{1}{2}$ baseline contribution.

B Applies the substitution correctly but evaluates $\int_0^2 1\,dx$ as 1 rather than 2.

C Attempts brute-force polynomial division, isolates the constant term $\frac{2}{3}$ and reports it without finishing.

E Brute force: integrates $\frac{x}{3} + \frac{2}{3}$ from 0 to 2 to get 2, then applies the $\frac{1}{2}$ factor inconsistently.

F Reaches $2I = 2$ via the symmetry pairing but reports the value of $2I$ instead of I.

Question 20

If $\log_a b = p/q$ in lowest terms, then $b^q = a^p$. By unique factorisation, a and b must be integer powers of a common integer $c \geq 2$. Each integer ≥ 2 has a unique *primitive base* c (not itself a perfect power), so $\log_a b$ is rational iff a and b share the same primitive base. Enumerate by primitive base c with $c^{p_{\max}} \leq 100$: valid pairs are (c^q, c^p) with $1 \leq q < p \leq p_{\max}$, giving $\binom{p_{\max}}{2}$. We get $c = 2$: $\binom{6}{2} = 15$; $c = 3$: $\binom{4}{2} = 6$; $c = 5, 6, 7, 10$: $\binom{2}{2} = 1$ each. (Bases $4, 8, 9$ are perfect powers, already counted; $c \geq 11$ non-power gives $c^2 > 100$.) Total $= 15 + 6 + 4 = 25$.

Distractor analysis:

A Demands $\log_a b$ be a positive integer rather than rational, missing pairs like $(4, 8)$ with $\log_4 8 = 3/2$. Counts $b = a^k$ for $k \geq 2$ over $a = 2, \ldots, 10$, giving $5+3+2+1+1+1+1+1+1 = 16$.

B Only considers primitive bases $c = 2, 3$, overlooking $c = 5, 6, 7, 10$: $15 + 6 = 21$.

C Restricts the primitive base to prime values, missing the composite primitive bases $c = 6, 10$: $15 + 6 + 1 + 1 = 23$.

D Excludes the pair $(10, 100)$ on the mistaken belief that $b = 100$ violates the bound.

F Treats $c = 4, 8, 9$ as additional primitive bases, double-counting: $25 + \binom{3}{2} + 1 + 1 = 30$.

G Counts ordered pairs without imposing $a < b$, doubling to 50.